

Beyond the Tao-Thouless limit of the fractional quantum Hall effect: spin chains and Fermi surface deformation

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Abstract. We discuss the relationship between the fractional quantum Hall effect in the vicinity of the thin-torus, a.k.a. Tao-Thouless (TT), limit and quantum spin chains. We argue that the energetics of fractional quantum Hall states in Jain sequence at filling fraction $\nu = p/(2p+1)$ (and $\nu = 1 - p/(2p+1)$) in the lowest Landau level is captured by $S = 1$ spin chains with p spins in the unit cell. These spin chains naturally arise at sub-leading order in $e^{-2\pi^2/L_1^2}$ which serves as an expansion parameter away from the TT limit ($L_1 \rightarrow 0$). We also corroborate earlier results on the smooth Fermi surface deformation of the gapless state at $\nu = 1/2$, interpolating between a state described by a critical $S = 1/2$ chain and the bulk.

1. Introduction

Ultra cold electrons in two dimensions in a strong perpendicular magnetic field, the quantum Hall (QH) system [1], exhibits a fascinating phase diagram including phases with fractionalized excitations and topological order [2, 3, 4, 5, 6]. Ever since its discovery three decades ago, the QH system has inspired a huge amount of experimental and theoretical effort, not least due to its richness in phenomenology and mathematical structure. New developments include the observation of the fractional quantum Hall effect in graphene [7] and ideas of applications in the context of topological quantum computing [8]. Moreover, it has been realized that the theoretical description of a system of rapidly rotating bosons is formally very similar to that of an electron gas in a magnetic field [9].

A key property of fractional quantum Hall (FQH) states is their topological order [10]. One consequence thereof is that their physical properties are insensitive to smooth deformations of the manifold on which we choose to study them. This fact has been exploited in a series of recent studies of the interacting many-body problem in the limit geometry of a thin torus, referred to as the Tao-Thouless (TT) limit [11, 12, 13, 14] (see also Refs. [15, 16, 17] for precursory studies and Ref. [22] for some recent related approaches).

In this proceeding we study the FQH system beyond the TT limit and argue that some characteristics that are not readily understood in the TT limit states are however manifest in the leading quantum fluctuations away from this limit. While several of the ideas presented here have been published earlier [11, 19, 20, 18], we corroborate these results and also present a number of new results.

We study the structure of excitation spectra of two-dimensional electrons in a magnetic field with Coulomb interactions as a function of the torus circumference, L_1 , and contrast the cases of even and odd denominator filling fractions. For $\nu = 1/2$ we interpret these results in terms of a deforming Fermi sea [11]. Then, by extending the results of Refs. [19, 20] for the $\nu = 1/3$ FQH state, we further establish that FQH systems in a Jain sequence[5], which has filling factor $\nu = p/(2p + 1)$ in the lowest Landau level can be described by $S = 1$ spin chains with p sites unit cell.

The remainder of this proceeding is organized as follows. In Section 2 we review the lattice describing interacting particles in a Landau level on the torus and present numerical spectra thereof as a function of the circumference, L_1 . In Section 3, we briefly motivate and restate known results about the energy spectra in the TT limit and then introduce spin models that provide an intuitive understanding of the numerical data away from this limit. In particular, these considerations shed light on why odd denominator fractions remain gapped while even denominator states tend to be gapless.

2. One-dimensional description and numerical simulations

We consider a model of N interacting electrons on a torus pierced by N_s flux quanta. In the Landau gauge, a complete basis of N_s degenerate single-particle states in the lowest Landau level, labeled by $k = 0, \dots, N_s - 1$, can be chosen as

$$\psi_k(x) = (\pi^{1/2} L_1)^{-1/2} \sum_{n=-\infty}^{\infty} e^{i(k_1 + nL_2)x_1} e^{-\frac{1}{2}(x_2 + k_1 + nL_2)^2}, \quad (1)$$

where L_i are the circumferences of the torus, x_i the corresponding coordinates, and $k_1 = 2\pi k/L_1$ the momentum along the L_1 -cycle. We have set the magnetic length $l_B \equiv \sqrt{\hbar/eB}$ equal to unity.

In this basis, any translation-invariant two-dimensional two-body interaction Hamiltonian assumes the following model on a one-dimensional discrete lattice,

$$\mathcal{H} = \sum_{k > |m|} \hat{V}_{km}, \quad \hat{V}_{km} \equiv V_{km} \sum_i c_{i+m}^\dagger c_{i+k}^\dagger c_{i+m+k} c_i, \quad (2)$$

where the matrix-element V_{km} specifies the amplitude for a process where particles with separation $k + m$ hop m steps to a separation $k - m$ (note that m can be negative). The number of the lattice sites is fixed by the area, $N_s = L_1 L_2 / 2\pi$. At the filling $\nu = p/q$ the Hamiltonian commutes with the center-of-mass magnetic translations [23] T_1 and T_2^q along the cycles, which implies, in particular, that the total momentum K_1 along the L_1 -cycle is conserved modulo N_s in this gauge. In this system (2), two conservation numbers are given as

$$T_1 : e^{i2\pi K_1} = \exp\left(i \frac{2\pi}{N_s} \sum_{j=1}^{N_s} j n_j\right), \quad T_2^q : e^{i2q\pi K_2/N_s}, \quad (3)$$

where $n_j \equiv c_j^\dagger c_j$ and $K_2 = 0, 1, \dots, N_s/q$.

We have calculated energy spectra of this system for $\nu = 1/2, 1/3, 2/5$ using exact diagonalization in subspaces labeled by (K_1, K_2) . Using these symmetries, the Hilbert space splits into $(N_s/q)^2$ sectors that are all of comparable size. This enable us to treat larger systems. The matrix elements of V_{km} are calculated assuming Coulomb interaction between the electrons. For the half-filled Landau level $\nu = 1/2$, we have calculated the energies up to $N_s = 22$ which is larger than the former analysis [11] (see Fig. 1). Increasing L_1 from the TT limit ($L_1 \rightarrow 0$), the first level-crossing in the ground state energy takes place at $L_1 \simeq 5.3$

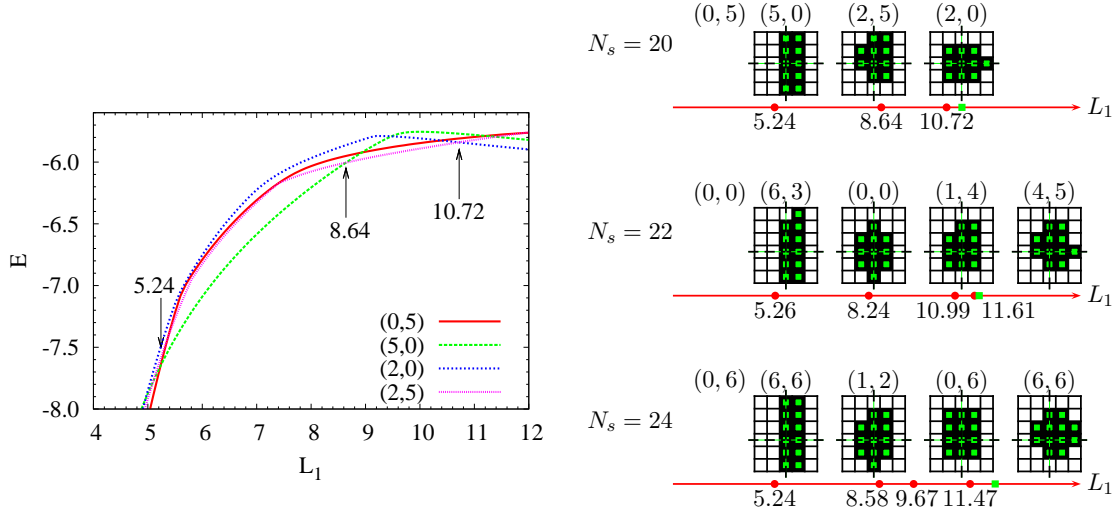


Figure 1. The left panel shows energy spectra of the Hamiltonian (2) with Coulomb potential for $\nu = 1/2$ with $N_s = 20$ as a function of L_1 . The corresponding phase diagram with “Fermi sea” representation of the gapless states labeled by quantum numbers (K_1, K_2) [11] is shown in the right panel along with analogous phase diagrams for $N_s = 22$ and $N_s = 24$. The initial phase transition at $L_1 \approx 5.2 - 5.3$ as well as the level crossings between the different Fermi seas are denoted by red circles while green boxes denote the symmetric points $L_1 = L_2$ which occur at $L_1 = 11.21$ ($N_s = 20$), $L_1 = 11.76$ ($N_s = 22$) and $L_1 = 12.28$ ($N_s = 24$).

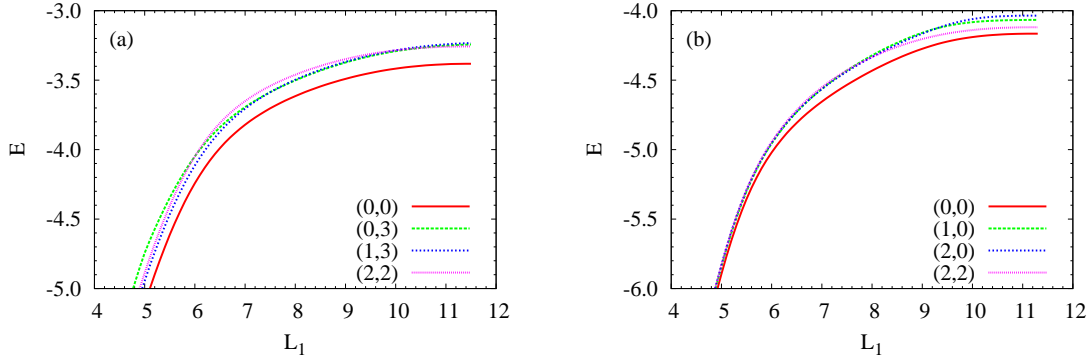


Figure 2. Energy spectra of the Hamiltonian (2) for (a) $\nu = 1/3$ with $N_s = 21$ and (b) $\nu = 2/5$ with $N_s = 20$. The energy gaps never close as functions of the circumference L_1 of the torus.

which corresponds to a phase transition from a charge-density-wave (CDW) to a gapless phase. Further level crossings occur among different gapless states, and finally L_1 arrives at the duality point $L_1 = L_2$ ($L_1 = 11.21$ for $N_s = 20$, $L_1 = 11.76$ for $N_s = 22$ and $L_1 = 12.28$ for $N_s = 24$). The spectrum is invariant under $L_1 \leftrightarrow L_2$ (although quantum numbers change) and hence we only present data for $L_1 \leq L_2$. We have calculated the energies of the $\nu = 1/3$ and $\nu = 2/5$ cases, and confirmed that energy gaps do not close as L_1 changes, and level-crossings occur only in excited states. This is consistent with the fact that these states have energy gaps in the bulk systems as is evident in the thin-torus limit (see below).

3. Beyond the Tao-Thouless limit: Effective spin chains

3.1. Tao-Thouless limit and leading quantum fluctuations

From the shape of the single particle states (1) it follows that the distance between adjacent states, the lattice constant $2\pi/L_1$, depends on L_1 while the extent of the wave functions is constant. For small L_1 the overlap between different single particle states decreases rapidly and the matrix elements V_{km} simplify considerably. As $L_1 \rightarrow 0$ one finds that

$$V_{km} \sim e^{-2\pi^2 m^2/L_1^2} V_{k0}, \quad (4)$$

thus the $m \neq 0$ terms are exponentially suppressed for generic interactions in this limit. The remaining ($m = 0$) problem is exactly solvable: ground states at any $\nu = p/q$ are gapped periodic crystals (with a unit cell of p electrons on q sites) and the fractionally charged excitations appear as domain walls between degenerate ground states. The fractal structure (a.k.a. Devils staircase) of the Abelian Haldane-Halperin hierarchy [3, 4] is manifest for generic convex two-body interactions and provides an explicit estimate for the relative stability of odd denominator states in excellent agreement with experiment [14, 28].

The leading fluctuations are the $m = 1$ terms describing correlated hopping of two particles one site each in opposite directions. In a series of earlier investigations, some of us have shown that these fluctuations ultimately lead to a melting of the gapped state for $\nu = 1/2$ resulting in a gapless state [11], while for $\nu = 1/3$ the excitation structure and gap is essentially unaltered, even for very strong fluctuations [19, 20]. Below, we discuss how the numerical results at finite L_1 can be interpreted in terms of effective spin chains and extend this picture to include the most prominent ($m = 1$) Jain series, $\nu = p/(2p + 1)$ and $\nu = (p + 1)/(2(p + 1) + 1)$. The strategy is as follows. First, we truncate the interaction to only include the leading $m = 1$ term, \hat{V}_{21} . This is especially sensible for a pseudo potential interaction where $V_{km} = P(k, m)e^{-2\pi^2(k^2+m^2)/L_1^2}$ with $P(k, m)$ being a polynomial¹ in k and m . In this case our expansion in $e^{-2\pi^2/L_1^2}$ is well controlled, and we expect it to capture the physics also for more general interactions. Second, we restrict the local Hilbert space by including only local configurations that correspond to the TT ground state and local applications of \hat{V}_{21} . This naturally leads to a spin chain representation of the Hamiltonian and enables us to deform it to connect to well studied spin chain models and obtain physical insights by drawing analogies between the physics of spin chains and FQH states.

3.2. $S = 1/2$ chain for $\nu = 1/2$

The electron system on a torus can be described in terms of $S = 1/2$ variables in the following way [11]: In the vicinity of the TT limit, the relevant physics is captured by a model containing the three dominant terms as

$$\mathcal{H}_t = \sum_i [V_{10}n_i n_{i+1} + V_{20}n_i n_{i+2} + V_{21}(c_{i+1}^\dagger c_i c_{i+2}^\dagger c_{i+3} + \text{H.c.})], \quad (5)$$

where $V_{10} > V_{20} > V_{21}$. In the TT limit, the ground state is the charge-density-wave (CDW) state where electrons are located on every two sites $|\cdots 010101010\cdots\rangle$, since the matrix elements of the electrostatic term V_{10} is dominant. Away from the TT limit, the competition between V_{10} , V_{20} and V_{21} can be included as an interaction of the local spin states $|10\rangle \rightarrow |\uparrow\rangle$ and $|01\rangle \rightarrow |\downarrow\rangle$ which means that $c_{2n}^\dagger c_{2n+1} \rightarrow S_n^+$, $c_{2n+1}^\dagger c_{2n} \rightarrow S_n^-$, $c_{2n}^\dagger c_{2n} \rightarrow 1/2 + S_n^z$, $c_{2n+1}^\dagger c_{2n+1} \rightarrow 1/2 - S_n^z$, where n is the index of the unit cell. The effective spin Hamiltonian is then given by the $S = 1/2$ XXZ chain,

$$\mathcal{H}_{XXZ} = \sum_n \frac{1}{2} (S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+) + \Delta S_n^z S_{n+1}^z, \quad (6)$$

¹ See Ref. [21] for explicit expressions.

where $\Delta = (2V_{20} - V_{10})/4V_{21}$. It is well known that this model has a phase transition from a ferromagnetic (CDW) phase to a gapless (Tomonaga-Luttinger liquid) phase at $\Delta = -1$. This describes the phase transition from the TT state to the gapless phase for the half-filled Landau level $\nu = 1/2$. The obtained solution corresponds to the (first) elongated Fermi seas in Fig. 1 appearing for $L_1 \gtrsim 5.3$.

3.3. $S = 1$ chain for $\nu = 1/3$

Next we consider spin-mapping for $\nu = 1/3$ state. In the TT limit, the ground state of the system is the three-fold degenerate CDW state $|\cdots \underline{010} \underline{010} \cdots\rangle$ where the underlines denote unit cells. To capture the effect of the leading quantum fluctuations, we introduce $S = 1$ spin variables as $|010\rangle \rightarrow |0\rangle$, $|100\rangle \rightarrow |+\rangle$, $|001\rangle \rightarrow |-\rangle$, where the operators are related as

$$c_{3n}^\dagger c_{3n+1} \rightarrow \frac{1}{\sqrt{2}} S_n^z S_n^+, \quad c_{3n+1}^\dagger c_{3n+2} \rightarrow \frac{1}{\sqrt{2}} S_n^+ S_n^z. \quad (7)$$

In this mapping, the three-fold degeneracy of the ground state is hidden. The degenerate states have different center-of-mass quantum numbers and there are no nonvanishing matrix elements between states in different momentum sectors. It follows that there are three distinct sectors which are described by the same effective Hamiltonian,

$$\mathcal{H}_{1/3} = \sum_n -\frac{V_{21}}{2} S_n^z S_n^+ S_{n+1}^z S_{n+1}^- + \text{H.c.} = \sum_n \frac{V_{21}}{2} S_n^+ S_{n+1}^- [1 - (S_n^z)^2] [1 - (S_{n+1}^z)^2] + \text{H.c.} \quad (8)$$

This model is given by four spin interactions or the XY model with projectors to the state $|\cdots 0000 \cdots\rangle$. Note that this Hamiltonian does not have the space inversion and spin reversal symmetries: the exchange process $|00\rangle \leftrightarrow |+-\rangle$ exists, but $|00\rangle \leftrightarrow |-+\rangle$ does not. The spin rotational symmetry is also broken. Actually, numerical analysis shows that this Hamiltonian well describes the FQH state around $L_1 \simeq 7l$, by checking the overlap between the truncated Hamiltonian and full Hamiltonian with Coulomb interaction and the Trugmann-Kivelson type potential [24] which provides an exact parent Hamiltonian for the Laughlin state. There is also a term given by a function $\sum_n f(S_{n+1}^z - S_n^z)$ which stems from the electrostatic terms, but this term does not alter any essential features of the present model (8) [31]. In Refs. [19, 20], this model with extra parameters

$$\mathcal{H}(\lambda, D) = \lambda \mathcal{H}_{1/3} + (1 - \lambda) \mathcal{H}_{\text{Hei}}(D), \quad (9)$$

$$\mathcal{H}_{\text{Hei}}(D) = \sum_n \{ \mathbf{S}_n \cdot \mathbf{S}_{n+1} + D(S_n^z)^2 \} \quad (10)$$

has been analyzed numerically to study adiabatic continuity from a conventional spin chain ($\lambda = 0$). For $\lambda = 0$, it is known that there is a phase transition at $D = D_c (\simeq 0.968)$ between the Haldane ($D < D_c$) and large- D ($D > D_c$) phases [29]. In Refs. [19, 20] it was shown that the ground state of the model (8) is smoothly connected both to the Haldane phase and the large- D phase without closing the energy gap under the change of parameters. This indicates that the spin model describing the $\nu = 1/3$ FQH state has the nature of these two phases simultaneously, and there are indeed some similarities between the $\nu = 1/3$ FQH effect and the $S = 1$ Haldane-gap state as noticed early on [26]. The reason for the absence of a phase transition is due to the breaking of the discrete symmetries. According to Ref. [30], Haldane and large- D phases are smoothly connected if the dihedral group, time reversal and parity symmetries are broken. Since the present model (9) with $\lambda \neq 0$ breaks these three symmetries, our result is consistent with this general argument. Most properties of a finite chain of this model (9) are similar to those of a usual large- D phase where no edge spins appear, however this is not inconsistent with the absence of a phase transition [30].

$c_{5n}^\dagger c_{5n+1}$	$2^{-1/2} T_n^z T_n^+$	$c_{5n+1}^\dagger c_{5n}$	$2^{-1/2} T_n^- T_n^z$
$c_{5n+1}^\dagger c_{5n+2}$	$-2^{-1/2} T_n^+ T_n^z$	$c_{5n+2}^\dagger c_{5n+1}$	$-2^{-3/2} T_n^z T_n^- S_n^- S_n^+$
$c_{5n+2}^\dagger c_{5n+3}$	$2^{-3/2} S_n^z S_n^+ T_n^+ T_n^-$	$c_{5n+3}^\dagger c_{5n+2}$	$2^{-1/2} S_n^- S_n^z$
$c_{5n+3}^\dagger c_{5n+4}$	$-2^{-1/2} S_n^+ S_n^z$	$c_{5n+4}^\dagger c_{5n+3}$	$-2^{-1/2} S_n^z S_n^-$

Table 1. Relationship between fermion and spin-1 operators for the $\nu = 2/5$ state.

3.4. Jain fractions

Let us now consider extensions of the above $S = 1$ mapping to other filling factors. As discussed by Jain [5], $\nu = p/(2mp + 1)$ FQH states may be described by the composite fermion picture, where $2m$ quantum flux are attached to non-interacting electrons of the p -th Landau level, and projection onto the lowest Landau level. For definiteness, we will consider the positive $m = 1$ Jain sequence $\nu = p/(2p + 1)$, $p > 0$. The results are easily transferred to the negative Jain series by noting that our formulation is particle-hole symmetric, hence the $\nu = p/(2p + 1)$ is equivalent to the $\nu = 1 - p/(2p + 1) = (p + 1)/(2(p + 1) - 1)$ case. This is in contrast to the wave function based approaches and the composite fermion phenomenology, which lack this symmetry. Before giving the general result, let us consider the $\nu = 2/5$ state ($p = 2$) as the simplest example. In the TT limit, the ground state is given by the CDW state with the configuration $|\cdots \underline{01010} \underline{01010} \cdots\rangle$. When \hat{V}_{21} and $\hat{V}_{21}^\dagger = \hat{V}_{2,-1}$ are applied to this CDW state several times, one finds that the number of electrons in each unit cell is always conserved, and the configuration $|\cdots 000 \cdots\rangle$ never appears. Therefore, configurations such as $|\cdots \underline{00100} \cdots\rangle$, $|\cdots \underline{00011} \cdots\rangle$, $|\cdots \underline{10001} \cdots\rangle$, $|\cdots \underline{11000} \cdots\rangle$ are absent in the truncated Hilbert space. This means that the $\nu = 2/5$ states can be mapped to two $S = 1$ variables by inserting 0 appropriately (between the two 1's) in each unit cell:

$$|01010\rangle \rightarrow |01[00]10\rangle \rightarrow |00\rangle, \quad (11)$$

$$|00110\rangle \rightarrow |00[10]10\rangle \rightarrow |-0\rangle, \quad (12)$$

$$|01100\rangle \rightarrow |01[01]00\rangle \rightarrow |0+\rangle. \quad (13)$$

The relationship between fermion and spin-1 operators (S_n^α , T_n^α) for the $\nu = 2/5$ state is summarized in Table 1. We now obtain an effective spin chain as

$$\mathcal{H}_{2/5} = \frac{V_{21}}{2} \sum_n [T_n^- T_n^z S_n^z S_n^+ + T_n^z T_n^- S_n^+ S_n^z - S_n^z S_n^- T_{n+1}^z T_{n+1}^+] + \text{H.c.} \quad (14)$$

Similarly, we can extend the $S = 1$ spin mapping for general p . For $p > 0$ cases, the configuration in the TT limit is given by $|\cdots \underline{0(10)_p} \cdots\rangle$, and we obtain $S = 1$ chains with p -sites unit cell (see also Fig. 3),

$$\mathcal{H}_{\frac{p}{2p+1}} = \frac{V_{21}}{2} \sum_n \left[\sum_{j=1}^{p-1} \left(S_{j,n}^- S_{j,n}^z S_{j+1,n}^z S_{j+1,n}^+ + S_{j,n}^z S_{j,n}^- S_{j+1,n}^+ S_{j+1,n}^z \right) - S_{p,n}^z S_{p,n}^- S_{1,n+1}^z S_{1,n+1}^+ \right] + \text{H.c.} \quad (15)$$

where $S_{j,n}^\alpha$ means $S = 1$ operators of j -th site in n -th unit cell. We have again ignored the contribution from the electrostatic terms, for simplicity. Note that this model includes the $\nu = 1/3$ model in eq. (8), but is not appropriate for the $\nu = 1/2$ state. Since this model also breaks the three discrete symmetries mentioned above, we expect that it has similar low-energy properties as $S = p$ quantum spin chains with Haldane gaps. For $p \leq -2$ cases, effective spin chains can be obtained by replacing p in eq. (15) by $|p| - 1$. The detailed derivation and further analysis of this model will be published elsewhere [31].

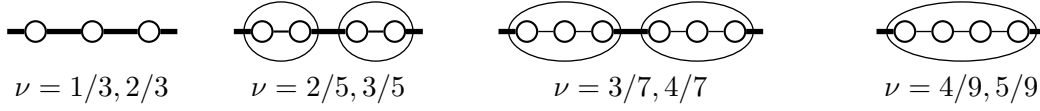


Figure 3. Effective $S = 1$ spin chains for $\nu = p/(2p + 1)$ ($p \geq 1$) FQH states given by p spins in a unit cell. Mapping of negative series can also be possible via particle-hole transformation.

4. Conclusion

We have studied the fractional quantum Hall (FQH) effect based on the one-dimensional description with torus boundary conditions and illustrated how energy spectra of interacting electrons in a magnetic field changes as functions of the circumference of the torus for various filling factors using exact diagonalization. In an effort to understand these results microscopically, we discussed spin chain mappings of the FQH states. By extending earlier results for the $\nu = 1/2$ and the $\nu = 1/3$ cases, we mapped the Jain sequences $\nu = p/(2p+1)$ and $\nu = (p+1)/(2(p+1)-1)$ to $S = 1$ chains with p spins in the unit cell that we conjecture to be gapped for all p . This further strengthens the known analogies between the FQH physics and the quantum spin chains. We have also studied the gapless state at $\nu = 1/2$ numerically. By going to slightly larger system sizes than before (from 10 to 12 particles), we confirmed that the evolution from the thin torus to the bulk is, after an initial first order transition that is accurately describe by an $S = 1/2$ spin chain model, smooth and described by a natural deformation of the quasi-particle Fermi sea.

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